

Section 2

DYNAMICS OF A PARTICLE

2.1 INTRODUCTION

2.1.1 In this section, the means of describing the motion of a particle will be investigated. The study of particle dynamics provides a logical starting point for the further examination of more complicated dynamical problems. In particular, several of the concepts developed in this section are used formally in the establishment of the theory of the dynamics of a rigid body. Further, in many physical problems of interest, the bodies involved may be considered to be particles; practical solutions are thus obtained without the need of more complex theory.

The development in this section is purely Newtonian. No energy considerations will be included in the discussion.

2.2 DEFINITION OF A PARTICLE

2.2.1 A particle is defined in terms of its following properties:

- The particle has a finite mass m .
- The particle has a measurable position, but no extension or dimension; it is analogous to a geometrical point in this respect.

2.3 POSITION, VELOCITY, ACCELERATION

2.3.1 Let the position of a moving particle be measured at some instant in a fixed Cartesian coordinate system (Fig. 17). In vector notation, the position of the particle is given by the position vector

$$\vec{r} = \vec{i}x + \vec{j}y + \vec{k}z$$

where $\bar{i}$, $\bar{j}$, and $\bar{k}$ are unit vectors along the X, Y, and Z axes, respectively.

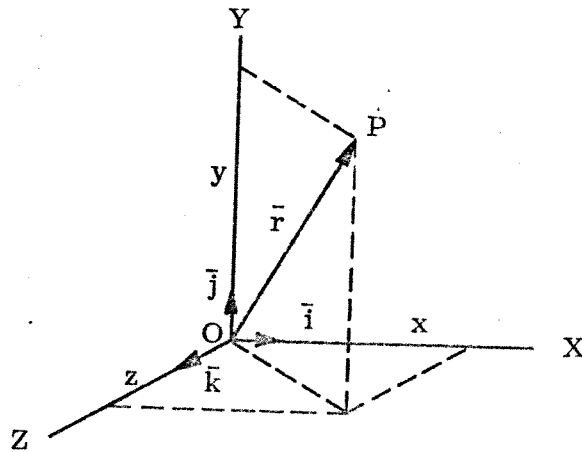


Fig. 17

The instantaneous velocity $\bar{v}$ is then defined as

$$\bar{v} = \frac{d\bar{r}}{dt} = \frac{d}{dt} (\bar{i}x + \bar{j}y + \bar{k}z) = (\bar{i} \frac{dx}{dt} + \bar{j} \frac{dy}{dt} + \bar{k} \frac{dz}{dt}) + (x \frac{d\bar{i}}{dt} + y \frac{d\bar{j}}{dt} + z \frac{d\bar{k}}{dt})$$

But, since the unit vectors are constant in magnitude and direction,

$$\frac{d\bar{i}}{dt} = \frac{d\bar{j}}{dt} = \frac{d\bar{k}}{dt} = 0$$

and

$$\bar{v} = \bar{i} \frac{dx}{dt} + \bar{j} \frac{dy}{dt} + \bar{k} \frac{dz}{dt}$$

The magnitude of the velocity is given by

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

and the velocity vector is tangent to the path of the moving particle at any point along the path.

The instantaneous acceleration $\bar{a}$ is defined as

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{d}{dt} \left(\bar{i} \frac{dx}{dt} + \bar{j} \frac{dy}{dt} + \bar{k} \frac{dz}{dt} \right) = \bar{i} \frac{d^2x}{dt^2} + \bar{j} \frac{d^2y}{dt^2} + \bar{k} \frac{d^2z}{dt^2}$$

The magnitude of the acceleration is given by

$$a = \sqrt{\left(\frac{d^2x}{dt^2} \right)^2 + \left(\frac{d^2y}{dt^2} \right)^2 + \left(\frac{d^2z}{dt^2} \right)^2}$$

In further discussion, frequent use will be made of dot notation, whereby the symbol $\dot{}$ is used to represent differentiation with respect to time. Thus,

$$\bar{v} = \dot{\bar{r}} \quad \text{and} \quad \bar{a} = \dot{\bar{v}} = \ddot{\bar{r}}$$

If the motion of the particle lies wholly in a plane, it may be convenient in the solution of a particular problem to use polar coordinates. We shall therefore investigate the representations of velocity and acceleration, as defined above, in terms of the polar coordinates r and θ (Fig. 18).

Let $\bar{r}_0$ and $\bar{\theta}_0$ be unit vectors in directions along and perpendicular to the position vector $\bar{r}$. At time t , the position of the particle is given by the position vector $\bar{r}(t)$. At a slightly later time $t + \Delta t$, the position of the particle is $\bar{r}(t + \Delta t)$. The displacement of the particle in the time increment Δt is therefore

$$\bar{r}(t + \Delta t) - \bar{r}(t)$$

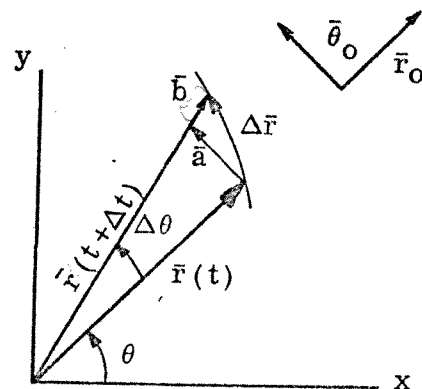


Fig. 18

which is the vector $\Delta \bar{r}$ in Fig. 18. The displacement $\Delta \bar{r}$ can be resolved into components $\bar{a}$ and $\bar{b}$ in both θ and r directions as in

$$\Delta \bar{r} = \bar{a} + \bar{b} = a \bar{\theta}_o + b \bar{r}_o = r \Delta \theta \bar{\theta}_o + \Delta r \bar{r}_o$$

Dividing by Δt , gives

$$\frac{\Delta \bar{r}}{\Delta t} = r \frac{\Delta \theta}{\Delta t} \bar{\theta}_o + \frac{\Delta r}{\Delta t} \bar{r}_o$$

and taking limits, we have

$$\bar{v} = \frac{d\bar{r}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \bar{r}}{\Delta t}$$

and

$$\bar{v} = \dot{r} \bar{r}_o + r \dot{\theta} \bar{\theta}_o \quad (2.1)$$

Having thus found the velocity vector in polar coordinates, we proceed to derive an expression for the acceleration, as in

$$\bar{a} = \frac{d\bar{v}}{dt} = \frac{d}{dt} (\dot{r} \bar{r}_o + r \dot{\theta} \bar{\theta}_o) = \ddot{r} \bar{r}_o + \dot{r} \dot{\bar{r}}_o + (\dot{r} \dot{\theta} + r \ddot{\theta}) \bar{\theta}_o + r \dot{\theta} \dot{\bar{\theta}}_o$$

Since the unit vectors $\bar{r}_o$ and $\bar{\theta}_o$ change their orientation as the particle moves, their derivatives do not, in general, equal zero.

To evaluate $\dot{\bar{r}}_o$, consider the infinitesimal displacement $\Delta\bar{r}_o$ (Fig. 19) which the unit vector $\bar{r}_o$ undergoes in an infinitesimal increment of time Δt :

$$\Delta\bar{r}_o = |\bar{r}_o| \Delta\theta \bar{\theta}_o = 1 \cdot \Delta\theta \bar{\theta}_o$$

$$\frac{\Delta\bar{r}_o}{\Delta t} = \frac{\Delta\theta}{\Delta t} \bar{\theta}_o$$

and, taking limits, we have

$$\dot{\bar{r}}_o = \dot{\theta} \bar{\theta}_o$$

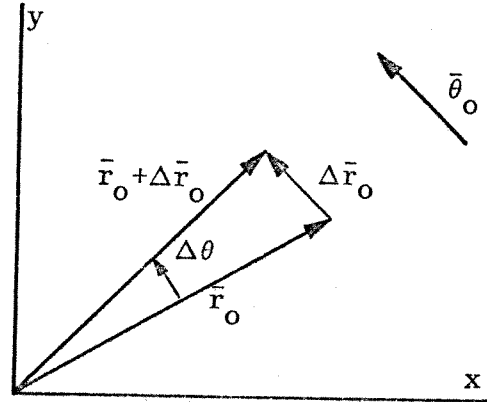


Fig. 19

To evaluate $\dot{\bar{\theta}}_o$, consider the infinitesimal displacement $\Delta\bar{\theta}_o$ (Fig. 20) which the unit vector $\bar{\theta}_o$ undergoes in an infinitesimal increment of time Δt :

$$\Delta\bar{\theta} \cong |\bar{\theta}_o| \Delta\theta (-\bar{r}_o) = -1 \cdot \Delta\theta \bar{r}_o$$

$$\frac{\Delta\bar{\theta}_o}{\Delta t} \cong -\frac{\Delta\theta}{\Delta t} \bar{r}_o$$

and, taking limits, we obtain

$$\dot{\bar{\theta}}_o = -\dot{\theta} \bar{r}_o$$

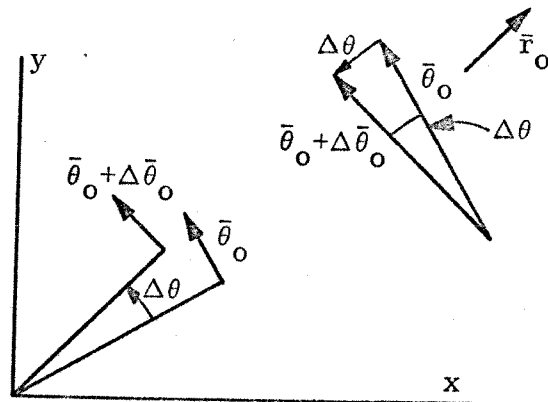


Fig. 20

Substituting for $\dot{\bar{r}}_o$ and $\dot{\bar{\theta}}_o$ in the expression for $\bar{a}$, we have

$$\bar{a} = \ddot{r} \bar{r}_o + \dot{r} \dot{\theta} \bar{\theta}_o + (\dot{r} \dot{\theta} + r \ddot{\theta}) \bar{\theta}_o - r \dot{\theta}^2 \bar{r}_o$$

$$= (\ddot{r} - r \dot{\theta}^2) \bar{r}_o + (2\dot{r} \dot{\theta} + r \ddot{\theta}) \bar{\theta}_o$$

(2.2)

Thus, in Eqs. (2.1) and (2.2), we now have expressions for the velocity and acceleration of the particle in polar coordinates. See paragraph 2.7 for an application of these equations.

Similar considerations would lead to representations of these vectors in any other two-or-three-dimensional coordinate system.

2.4 NEWTON'S LAWS

2.4.1 The basis for the mechanics presented in this report is that expressed in the form of Newton's "laws" of motion, first enunciated in the Principia Mathematica:

- I. Every particle continues in its state of rest or uniform motion in a straight line, unless it is acted upon by some exterior force.
- II. The rate-of-change of momentum of a particle is proportional to the force impressed upon it, and is in the direction in which the force is acting.
- III. To every action, there is an equal and oppositely directed reaction.

It is not within the province of this discussion to investigate formally the interpretation of these laws, but a few comments on their meaning are appropriate.

The first law depends for its meaning on an understanding of the concept of force. One has an intuitive feeling for the meaning of the word force in terms of a push or pull, and, from this sort of notion, a qualitative idea of the sense of the first law can be obtained. However, no quantitative knowledge can be gained from the first law, except possibly the definition of zero force; that is, no force is acting when the ideal state described exists. The first law is therefore a kind of qualitative introduction to the second law.

The second law expressed the following mathematical relationship between the force acting on a particle and the mass of the particle:

$$\bar{F} = \frac{d}{dt} (m \bar{v})$$

For most purposes outside the field of relativity, mass is constant; thus, the above equation reduces to

$$\bar{F} = m \bar{a}$$

In order for such a relationship to express something meaningful, we must not only know what force is and how to measure it, we must also know the meaning of the mass m of the particle. Newton considered the notion of mass, like that of force, as more or less intuitively known. Thus, the inherent difficulties in the concepts of force and mass remained largely unresolved until the appearance of Ernst Mach's Mechanics at the end of the nineteenth century. Mach provided a sound and logical theoretical basis which replaces Newton's laws as a foundation of mechanics, and which leads to the results which Newton sought to draw from these laws. Mach's development provides postulational definitions of mass and force which lead to the deduction of Newton's third law.

A more detailed discussion of this subject is not pertinent here; suffice it to say that classical Newtonian mechanics does have a logical theoretical basis, although the formal development of that basis did not come until the end of the 19th century.

2.5 THE EQUATION OF MOTION

2.5.1 Having reassured ourselves that Newtonian mechanics has a logical basis, we shall utilize the relationship (paragraph 2.3.1) between the behavior of a particle to the force acting upon it through a second-order vector differential equation:

$$\bar{F} = m \ddot{\bar{r}} \quad (2.3)$$

where $\bar{F}$ is the resultant force acting on the particle; i.e., the vector sum of all the individual forces. Equation (2.3) is called the equation of motion of the particle. Substitution of a particular functional form of $\bar{F}$ (where $\bar{F}$ is, in general, a function of position, velocity, and time) in Eq. (2.3) yields a set of three scalar second-order differential equations in terms of the three components of $\bar{r}$. Solution of these equations of motion, subject to the constraints of given initial conditions, gives the position and velocity of the particle at any time.

In practice, a form of F is guessed at, and, if the solution of the resulting equation yields answers which are in agreement with both past and future observations, the assumed description of the force is considered to be accurate.

Since the equation of motion is of the second order, the solution always involves two arbitrary constants which must be specified in order to provide the final solution in any given problem. Knowledge of the position and velocity at a given instant is sufficient to fix these constants; these constants are the usual boundary conditions involved in particle dynamics.

2.6 MOTION UNDER A CONSTANT FORCE

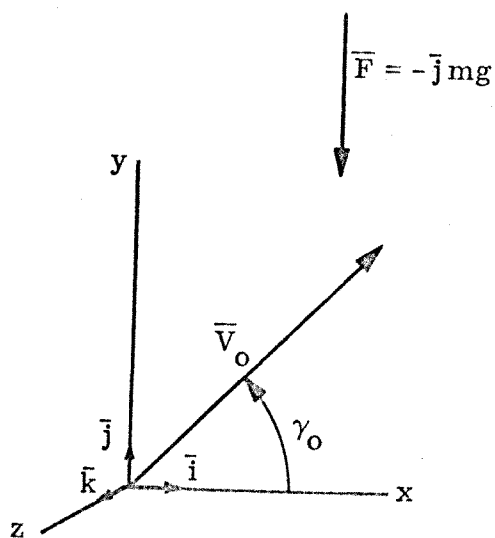


Fig. 21

2.6.1 As an elementary example of the method of solution of the equation of motion, consider the motion of a particle under a constant force of magnitude mg acting in the negative y direction of an arbitrary xyz coordinate system (Fig. 21). Further, let us assume that the initial position of the particle is at the origin and that the initial velocity vector of the particle is of magnitude V_0 and lies in the xy plane at an angle γ_0 above the xz plane.

With $\bar{F} = -mg\bar{j}$, the equation of motion is $-mg\bar{j} = \ddot{\bar{r}}$. Writing the three scalar component differential equations, we have

$$\begin{aligned} m\ddot{x} &= 0 & \ddot{x} &= 0 \\ m\ddot{y} &= -mg & \ddot{y} &= -g \\ m\ddot{z} &= 0 & \ddot{z} &= 0 \end{aligned}$$

Integrating each of these equations twice yields

For the First Integration

$$\dot{x} = V_{xo}$$

$$\dot{y} = V_{yo} - gt$$

$$\dot{z} = V_{zo}$$

For the Second Integration

$$x = x_o + V_{xo}t$$

$$y = y_o + V_{yo}t - \frac{gt^2}{2}$$

$$z = z_o + V_{zo}t$$

where x_o , V_{xo} , y_o , V_{yo} , z_o , V_{zo} are constants of integration.

According to our initial conditions, $\bar{r}(0) = 0$; i.e., $x(0) = 0$, $y(0) = 0$, and $z(0) = 0$. Evaluation of the equations for x , y , and z at $t = 0$ yields $x(0) = x_o$, $y(0) = y_o$, and $z(0) = z_o$. Thus, $x_o = y_o = z_o = 0$.

The initial component values of the velocity vector are obtained from the initial conditions:

$$\dot{x}(0) = V_o \cos \gamma_o$$

$$\dot{y}(0) = V_o \sin \gamma_o$$

$$\dot{z}(0) = 0$$

Evaluation of the equations for $\dot{x}$, $\dot{y}$, and $\dot{z}$ at $t = 0$ yields $\dot{x}(0) = V_{xo}$, $\dot{y}(0) = V_{yo}$, $\dot{z}(0) = V_{zo}$. Thus $V_{xo} = V_o \cos \gamma_o$, $V_{yo} = V_o \sin \gamma_o$, and $V_{zo} = 0$.

Having thus evaluated all six constants of integration, we may write the parametric equations for the position and velocity of the particle at any time t , as follows:

$$\begin{aligned} x &= V_o t \cos \gamma_o & \dot{x} &= V_o \cos \gamma_o \\ y &= V_o t \sin \gamma_o - \frac{1}{2} g t^2 & \dot{y} &= V_o \sin \gamma_o - g t \\ z &= 0 & \dot{z} &= 0 \end{aligned}$$

Since $z = 0$ for all time, the motion of the particle takes place entirely within the xy plane, a fact which is intuitively obvious from the statement of the initial conditions.

To obtain an equation for the path of the particle in space, we may eliminate t from the position equations. Solving the x equation for t , we get

$$t = \frac{x}{V_o \cos \gamma_o}$$

Substituting for t in the y equation,

$$\begin{aligned} y &= \frac{V_o x \sin \gamma_o}{V_o \cos \gamma_o} - \frac{1}{2} g \frac{x^2}{V_o^2 \cos^2 \gamma_o} \\ &= x \tan \gamma_o - \frac{1}{2} \frac{g x^2}{V_o^2 \cos^2 \gamma_o} \end{aligned}$$

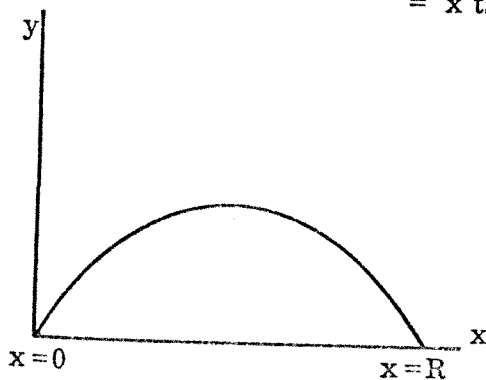


Fig. 22

This equation represents a parabola, as shown in Fig. 22. The range R of the particle is given by the solution of the above equation when $y = 0$ (of course, the other solution is given by $x = 0$, yielding the initial point of the particle).

$$0 = R \tan \gamma_0 - \frac{g R^2}{2 V_0^2 \cos^2 \gamma_0}$$

$$\tan \gamma_0 = \frac{g R}{2 V_0^2 \cos^2 \gamma_0}$$

Solving for R , we get

$$R = \frac{2 V_0^2 \cos^2 \gamma_0 \tan \gamma_0}{g} = \frac{2 \sin \gamma_0 \cos \gamma_0 V_0^2}{g} = \frac{V_0^2 \sin 2\gamma_0}{g}$$

It is seen that for a given launch velocity V_0 , maximum range will be obtained at a launch angle γ_0 of 45° , in which case, $\sin 2\gamma_0 = 1$, and

$$R_{\max} = \frac{V_0^2}{g}$$

It is clear that the above example represents a rough approximation to a trajectory computation, an approximation based on the assumption of a flat earth, the existence of a constant gravitational acceleration $-g$, and the neglect of aerodynamics and other forces.

This example, though elementary in nature, has served to illustrate a method by which the equation of motion is solved, and the role which the initial conditions play in arriving at an exact solution to a particular problem. We shall now consider a more complex example; namely, motion under a central force.

2.7 MOTION UNDER A CENTRAL FORCE.

2.7.1 A central force is defined as a force whose direction is along the position vector $\bar{r}$ of the particle, and whose magnitude is a function only of the distance r between the particle and the origin.

Thus, the general central force may be represented as follows:

$$\bar{F} = F(r)\bar{r}_0$$

where $\bar{r}_0$ is a unit vector along the radius vector as explained in paragraph 2.3. It is easily seen that all motion under a central force must be in a plane. This plane is defined by the initial force vector, which acts along the initial radius vector, and the initial velocity vector. Since the force always acts along the instantaneous radius vector, there is never a force component perpendicular to this plane of motion, and, therefore, no component of acceleration normal to the plane. Thus, as we found in the previous example, the motion of the particle lies entirely in this plane.

Description of this motion is most easily accomplished in terms of polar coordinates, and the results of our computation of velocity and acceleration vectors in polar coordinates, Eqs. (2.1) and (2.2), will be of value here. Writing the equation of motion in polar coordinates gives

$$\bar{F} = F(r)\bar{r}_0 = m\bar{a} = m \left[(\ddot{r} - r\dot{\theta}^2)\bar{r}_0 + (2\dot{r}\dot{\theta} + r\ddot{\theta})\bar{\theta}_0 \right]$$

Writing the scalar component equations, we have

$$F_r = F(r) = m(\ddot{r} - r\dot{\theta}^2)$$

$$F_\theta = 0 = m(2\dot{r}\dot{\theta} + r\ddot{\theta})$$

First, solving the differential equations in θ yields

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

$$\frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) = 0$$

$$r^2 \dot{\theta} = k \tag{2.4}$$

A physical meaning is readily attached to this result: In an infinitesimal interval of time Δt , the radius vector $\bar{r}$ sweeps out the area of the triangle shown in Fig. 23, $\frac{1}{2} r \cdot r \Delta \theta$. The rate at which area is being swept out is given by

$$\lim_{\Delta t \rightarrow 0} \frac{1}{2} r^2 \frac{\Delta \theta}{\Delta t} = \frac{1}{2} r^2 \dot{\theta}$$

This rate is referred to as the areal velocity. Therefore, in the case of a central force, what we have shown is that the areal velocity remains constant.

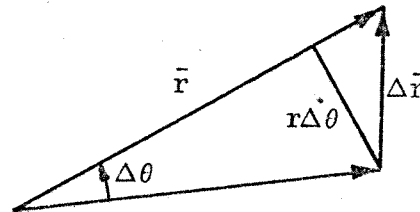


Fig. 23

The next step in our investigation is to eliminate time from our parametric equations, yielding a second-order differential equation in r and θ alone. It will be recalled that in the previous example (paragraph 2.6.1) this step was taken only after integration of the equations of motion was complete. Solution of the current example, however, is more easily accomplished if we take this course of action now.

From Eq. (2.4), we get

$$\dot{\theta} = \frac{k}{r^2} \quad (2.5)$$

Substituting this result for $\dot{\theta}$ into the r -component differential equation, we have

$$F(r) = m(\ddot{r} - r \frac{k^2}{r^4}) = m(\ddot{r} - \frac{k^2}{r^3}) \quad (2.6)$$

Next, changing our independent variable from t to θ and representing differentiation with respect to θ by the symbol $'$ yields

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt} = r' \dot{\theta} = r' \frac{k}{r^2} \quad [\text{from Eq. (2.5)}]$$

$$\ddot{r} = \frac{dr}{dt} = \frac{dr'}{dt} \frac{k}{r^2} + r' k \frac{d}{dt} \left(\frac{1}{r^2} \right)$$

$$\ddot{r} = \frac{dr'}{d\theta} \frac{d\theta}{dt} \frac{k}{r^2} + r' k \left(\frac{-2}{r^3} \right) \frac{dr}{d\theta} \frac{d\theta}{dt}$$

$$\ddot{r} = r'' \dot{\theta} \frac{k}{r^2} - \frac{2r' k}{r^3} r' \dot{\theta}$$

Substituting for $\dot{\theta}$ from Eq. (2.5), we have

$$\ddot{r} = r'' \frac{k}{r^2} \cdot \frac{k}{r^2} - \frac{2r' k}{r^3} r' \frac{k}{r^2}$$

$$\ddot{r} = \frac{k^2}{r^4} r'' - \frac{2k^2}{r^5} r'^2$$

Substituting for $\ddot{r}$ in Eq. (2.6) yields

$$\frac{F(r)}{m} = \frac{k^2}{r^4} r'' - \frac{2k^2}{r^5} r'^2 - \frac{k^2}{r^3} \quad (2.7)$$

We now introduce a change of variable by making the substitution $u = \frac{1}{r}$. Thus,

$$F(r) = F\left(\frac{1}{u}\right) = f(u)$$

$$r' = \frac{dr}{d\theta} = \frac{d}{d\theta} \left(\frac{1}{u} \right) = -\frac{u'}{u^2}$$

$$r'' = -\frac{u^2 u'' - 2uu'^2}{u^4} = \frac{2u'^2}{u^3} - \frac{u''}{u^2}$$

Substituting for r , r' , r'' , and $F(r)$ in terms of u in Eq. (2.7),

$$\frac{f(u)}{m} = k^2 u^4 \left(\frac{2u'^2}{u^3} - \frac{u''}{u^2} \right) - 2k^2 u^5 \left(\frac{-u'}{u^2} \right)^2 - k^2 u^3$$

$$\frac{f(u)}{m} = 2k^2 u u'^2 - k^2 u^2 u'' - 2k^2 u u'^2 - k^2 u^3$$

$$-\frac{f(u)}{m} = k^2 u^2 u'' + k^2 u^3$$

Finally, dividing both sides of the above equation by $k^2 u^2$, we get

$$u'' + u = -\frac{f(u)}{mk^2 u^2} \quad (2.8)$$

We have thus obtained a second order differential equation, independent of time, which describes the motion of a particle under any central force. A solution of this equation awaits a particular force formulation, $f(u)$.

2.8 MOTION UNDER AN INVERSE SQUARE FORCE LAW.

2.8.1 As a particular example of some importance, we shall consider a force of attraction which obeys the inverse square law; that is,

$$\overline{F}(r) = \frac{-c}{r^2} \overline{r}_0$$

$$F(r) = -\frac{c}{r^2}$$

$$f(u) = -cu^2$$

Substituting this value of $f(u)$ into Eq. (2.8) yields

$$u'' + u = \frac{c}{mk^2}$$

$$u'' + u - \frac{c}{mk^2} = 0$$

Let $\omega = u - \frac{c}{mk^2}$, and it follows that $\omega'' = u''$. Making the indicated substitution in the above equation, we get

$$\omega'' + \omega = 0$$

The general solution to this simple second-order differential equation may be written

$$\omega = A \cos (\theta - \alpha)$$

where A and α are constants of integration.

Since $\omega = u - \frac{c}{mk^2}$,

$$u = A \cos (\theta - \alpha) + \frac{c}{mk^2}$$

Substituting for u in terms of r ($r = 1/u$), we have

$$r = \frac{1}{A \cos (\theta - \alpha) + \frac{c}{mk^2}}$$

We now define two new constants, ϵ and p , in terms of the other constants as follows:

$$p = \frac{mk^2}{c}$$

$$\epsilon = Ap = \frac{mk^2 A}{c}$$

We then have

$$r = \frac{1}{A \cos(\theta - \alpha) + \frac{1}{p}} = \frac{p}{1 + Ap \cos(\theta - \alpha)}$$

and finally,

$$r = \frac{p}{1 + \epsilon \cos(\theta - \alpha)} \quad (2.9)$$

This expression may be recognized as the equation of a conic section in polar coordinates, with the origin at one of the foci. ϵ is the eccentricity of the orbit; if $\epsilon < 1$, the path is an ellipse; if $\epsilon = 1$, the path is a parabola; and if $\epsilon > 1$, the path is a hyperbola. The quantity p is the semi latus rectum. The angle θ is the polar angle, and α is the angle between the polar axis ($\theta = 0$) and the transverse axis of the conic (Fig. 24).

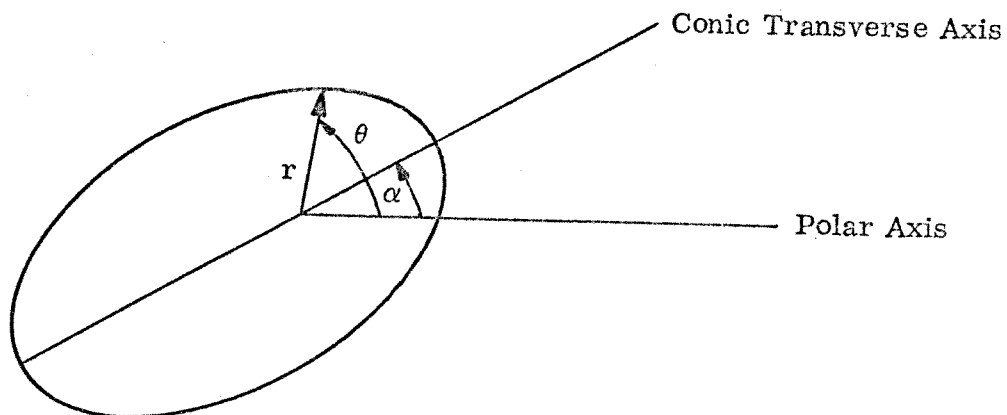


Fig. 24

For any particular problem, evaluation of the constants A , α , k and c depends on initial conditions of position (r_o, θ_o) and velocity $(\dot{r}_o, \dot{\theta}_o)$.

We may utilize the above result to obtain an expression which includes time. For simplicity, let us assume that the conic section representing the path of the particle is oriented with its axis along the polar axis. Equation (2.9) then becomes

$$r = \frac{p}{1 + \epsilon \cos \theta}$$

Differentiating with respect to θ , we have

$$\frac{dr}{d\theta} = \frac{-p(-\epsilon \sin \theta)}{(1 + \epsilon \cos \theta)^2} = \frac{\epsilon p \sin \theta}{(p/r)^2} = \frac{r^2 \epsilon \sin \theta}{p}$$

$$\dot{r} = \frac{dr}{d\theta} \dot{\theta} = \frac{r^2 \epsilon \sin \theta}{p} \cdot \frac{k}{r^2} \quad [\text{from Eq. (2.5)}]$$

$$\dot{r} = \frac{k\epsilon \sin \theta}{p}$$

Since

$$r = \frac{p}{1 + \epsilon \cos \theta} \quad \text{and} \quad \cos \theta = \frac{1}{\epsilon} \left(\frac{p}{r} - 1 \right)$$

$$\sin \theta = \pm \sqrt{1 - \frac{1}{\epsilon^2} \left(\frac{p}{r} - 1 \right)^2} = \pm \frac{1}{\epsilon} \sqrt{\epsilon^2 - \left(\frac{p}{r} - 1 \right)^2}$$

$$\dot{r} = \pm \frac{k}{p} \sqrt{\epsilon^2 - \left(\frac{p}{r} - 1 \right)^2}$$

Having thus eliminated the variable θ , we proceed to find t by integration, as follows:

$$\begin{aligned} dt &= \pm \frac{p}{k} \frac{dr}{\sqrt{\epsilon^2 - \left(\frac{p}{r} - 1\right)^2}} = \pm \frac{p}{k} \frac{r dr}{\sqrt{\epsilon^2 r^2 - p^2 + 2pr - r^2}} \\ &= \pm \frac{p}{k} \frac{r dr}{\sqrt{(\epsilon^2 - 1)r^2 + 2pr - p^2}} \\ t &= t_0 \pm \frac{p}{k} \int_{r_0}^r \frac{r dr}{\sqrt{(\epsilon^2 - 1)r^2 + 2pr - p^2}} \end{aligned}$$

The above integral is readily evaluated, and, with the specification of initial conditions t_0 and r_0 , an explicit relationship between r and t is obtained.

2.9 KEPLER'S LAWS.

2.9.1 In the early part of the seventeenth century, Kepler stated three laws governing the motion of a planet around the sun:

- I. Each planet moves in an ellipse about the sun, with the sun at one of the foci of the ellipse.
- II. The areas swept out in equal times by the radius vector from the sun to any planet are equal.
- III. The cubes of the major axes of the various planetary ellipses are directly proportional to the squares of the corresponding periods.

Newton later showed that Kepler's three laws are equivalent to a single simple law, as in the following argument.

Kepler's second law states that the rate-of-change of the areal velocity (paragraph 2.7) is zero; that is,

$$\frac{d}{dt} \left(\frac{d\bar{A}}{dt} \right) = 0$$

Considering the planet as a particle, the equation of motion of a planet becomes

$$\bar{F} = m \ddot{\bar{r}}$$

Taking the cross product of both sides of the above equation with $\bar{r}$, we have

$$\bar{r} \times \bar{F} = \bar{r} \times m \ddot{\bar{r}} = m \bar{r} \times \ddot{\bar{r}} = m \frac{d}{dt} (\bar{r} \times \dot{\bar{r}})$$

since

$$\frac{d}{dt} (\bar{r} \times \dot{\bar{r}}) = \bar{r} \times \ddot{\bar{r}} + \dot{\bar{r}} \times \dot{\bar{r}} = \bar{r} \times \ddot{\bar{r}} + 0$$

Expressing $\dot{\bar{r}}$ in polar coordinates,

$$\bar{r} \times \dot{\bar{r}} = \bar{r} \times (\dot{r} \bar{r}_o + r \dot{\theta} \bar{\theta}_o)$$

$$\bar{r} \times \dot{\bar{r}} = 0 + r \dot{\theta} (\bar{r} \times \bar{\theta}_o) = r^2 \dot{\theta} (\bar{r}_o \times \bar{\theta}_o) = 2 \frac{d\bar{A}}{dt}$$

where $\frac{d\bar{A}}{dt}$ is the areal velocity. Thus,

$$\bar{r} \times \bar{F} = 2m \frac{d}{dt} \left(\frac{d\bar{A}}{dt} \right)$$

But Kepler's second law states that

$$\frac{d}{dt} \left(\frac{d\bar{A}}{dt} \right) = 0$$

therefore, $\bar{r} \times \bar{F} = 0$.

If this relation is true in general, $\bar{r}$ and $\bar{F}$ must be colinear; and, therefore, $\bar{F}$ is a central force.

Kepler's first law states that, for any planet,

$$r = \frac{p}{1 + \epsilon \cos \theta}$$

which is the equation of an ellipse in polar coordinates.

If we let $u = \frac{1}{r}$, we have

$$u = \frac{1 + \epsilon \cos \theta}{p}$$

Differentiating twice with respect to θ ,

$$u'' = - \frac{\epsilon \cos \theta}{p}$$

Thus,

$$u'' + u = - \frac{\epsilon \cos \theta}{p} + \frac{1 + \epsilon \cos \theta}{p} = \frac{1}{p}$$

We have seen in section 2.7 that the general differential equation in u and θ for any central force [Eq. (2.8)] is

$$u'' + u = - \frac{f(u)}{mk^2 u^2}$$

Thus,

$$\frac{1}{p} = - \frac{f(u)}{mk^2 u^2}$$

$$f(u) = - \frac{mk^2 u^2}{p}$$

$$F(r) = - \frac{mk^2}{pr^2}$$

Thus for any planet, the force is attractive, with a magnitude inversely proportional to the planet's distance from the sun.

The period T of any planet around its ellipse is given by

$$T = \frac{A}{\frac{dA}{dt}}$$

where A is the area enclosed by the ellipse and $\frac{dA}{dt}$ is the magnitude of the previously defined areal velocity, which we have stated to be constant. Analytic geometry gives us the area of an ellipse in terms of its semimajor a and semiminor b axes.

$$A = \pi ab \quad \text{and} \quad b = a \sqrt{1 - \epsilon^2}$$

where ϵ is the eccentricity. Therefore,

$$T = \frac{\pi ab}{\frac{dA}{dt}} = \frac{2\pi a^2 \sqrt{1 - \epsilon^2}}{k}$$

where k is defined as in Eq. (2.5) to be twice the areal velocity. The equation of the ellipse we have seen to be

$$r = \frac{p}{1 + \epsilon \cos \theta}$$

r is at a maximum when $\theta = 0$ and is at a minimum when $\theta = \pi$. Thus,

$$r_{\max} = \frac{p}{1 + \epsilon} \quad \text{and} \quad r_{\min} = \frac{p}{1 - \epsilon}$$

$$2a = r_{\max} + r_{\min} = \frac{p}{1 + \epsilon} + \frac{p}{1 - \epsilon}$$

$$2a = \frac{2p}{1 - \epsilon^2}$$

$$1 - \epsilon^2 = \frac{p}{a} \quad \text{and} \quad \sqrt{1 - \epsilon^2} = \frac{\sqrt{p}}{\sqrt{a}}$$

Substituting for $\sqrt{1 - \epsilon^2}$ in the expression for T , we have

$$T = \frac{2\pi a^{3/2} p^{1/2}}{k}$$

In section 2.8, the constant p was defined as

$$P = \frac{mk^2}{c}$$

where c is the constant of proportionality of the inverse-square force law. Thus,

$$T = \frac{2\pi a^{3/2} \sqrt{m} k}{\sqrt{c} k}$$

$$T = \frac{2\pi a^{3/2} \sqrt{m}}{\sqrt{c}} \quad (2.10)$$

Since

$$T^2 = \frac{4\pi^2 m a^3}{c}$$

for each planet, Kepler's third law may be deduced from the other two. Thus, Kepler's three laws are equivalent to the one statement: The force relating the motion of each of the planets about the sun (origin) is given by

$$\bar{F} = -\frac{km}{r^2} \bar{r}_o$$

This reduction of Kepler's three empirical laws to a single statement concerning the force acting on each planet led Newton to his formulation of the law of universal gravitation.

2.10 ROTATIONAL MOTION AND ANGULAR VELOCITY

2.10.1 Consider a particle moving in a circular arc: In a time increment Δt , the particle will have moved through an angle $\Delta\phi$. The angular velocity $\bar{\omega}$ is defined as a vector with magnitude

$$\frac{d\phi}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta\phi}{\Delta t}$$

with direction normal to the plane of motion (shown as positive in Fig. 25).

Since $\bar{\omega}$ is a vector quantity, several simultaneous angular motions can be composed by the law governing composition of vectors:

$$\bar{\omega} = \bar{\omega}_1 + \bar{\omega}_2 + \dots + \bar{\omega}_n$$

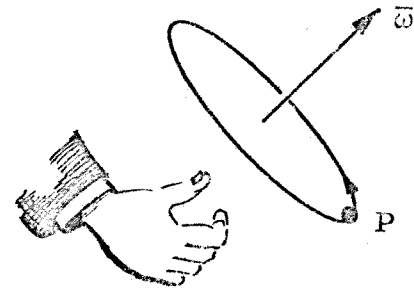


Fig. 25

This definition, though somewhat artificial, proves of great value in mechanics, particularly in the theory of dynamics of a rigid body.

We now proceed to find the relationship between the linear velocity $\bar{v}$ and the angular velocity $\bar{\omega}$. In Fig. 26, OB is the axis of rotation. The radius of the circular of motion BP is equal to $r \sin \theta$. The linear velocity, then, has a magnitude of $BP \phi = r \phi \sin \theta = r \omega \sin \theta$. If point O is taken as the origin of a coordinate system, $\bar{r}$ is the position vector of the particle. Then

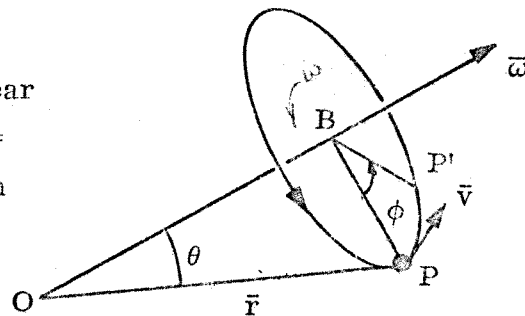


Fig. 26

$$|\bar{v}| = |\bar{r}| |\bar{\omega}| \sin \theta$$

and the direction of $\bar{v}$ is perpendicular to the plane defined by the vectors $\bar{r}$ and $\bar{\omega}$. We can thus write

$$\bar{v} = \bar{\omega} \times \bar{r}$$

which relates the angular velocity to the linear velocity.

2.11 NON-INERTIAL COORDINATE SYSTEMS

2.11.1 All the work done to this point has been based on the implicit assumption that the vectors $\bar{v}$ and $\bar{a}$ have been measured in a fixed coordinate system of some sort. In the real world, however, the concept of a fixed coordinate system is harder to visualize and impossible to find. The most nearly fixed reference frame susceptible to description is one oriented with respect to the average positions of the "fixed" stars. This system is generally called the primary inertial system.

Newton's laws, and the equations of motion derived from them, do not generally hold true in a reference frame moving with respect to the primary inertial system. We can show, however, that there is a type of moving system in which the equations of motion hold true. Suppose we have a reference system (X' , Y' , Z') moving with respect to the primary inertial system (X , Y , Z) at a constant velocity $\bar{v}_0$ without rotation. The position of a point in the moving system can then be written in terms of the primary inertial system: Thus,

$$\bar{r} = \bar{r}' + \bar{r}_0 + \bar{v}_0 t$$

where $\bar{r}_0$ is the initial position of the origin of the X' , Y' , Z' frame. Differentiating twice with respect to time,

$$\dot{\bar{r}} = \dot{\bar{r}}' + \bar{v}_0$$

$$\ddot{\bar{r}} = \ddot{\bar{r}}'$$

Thus, the equation of motion remains invariant with a change of coordinate system, and we have

$$\bar{F} = m\ddot{\bar{r}} = m\ddot{\bar{r}}'$$

Any such system moving with constant velocity with respect to the primary inertial system is called a secondary inertial system, and we have seen that the Newtonian equations of motion preserve their form in all such inertial systems.

Reference frames which are clearly noninertial may be considered inertial for a specific problem when the variation of the true motion, as measured in the noninertial frame from the Newtonian-derived motion, is small enough to be ignored. For many problems, on the other hand, this variation cannot be overlooked, requiring us to extend our concepts of velocity and acceleration to noninertial reference frames.

First, we shall consider a more general problem: Suppose we have two reference frames and that each possesses motion relative to the other. For convenience, we call one the reference frame and the other, the moving frame; however, we must remember that neither is assumed to be fixed in an inertial sense. An observer stationed on the moving frame measures the time rate of change of a vector quantity $\bar{g}$ and obtains a value $\frac{d\bar{g}}{dt}$. The question can then be asked: What will be the time rate of change of $\bar{g}$ measured by an observer stationed on the reference frame?

To answer this question, let us consider the frame (X, Y, Z) in motion with respect to a reference frame (X_0, Y_0, Z_0) (Fig. 27).

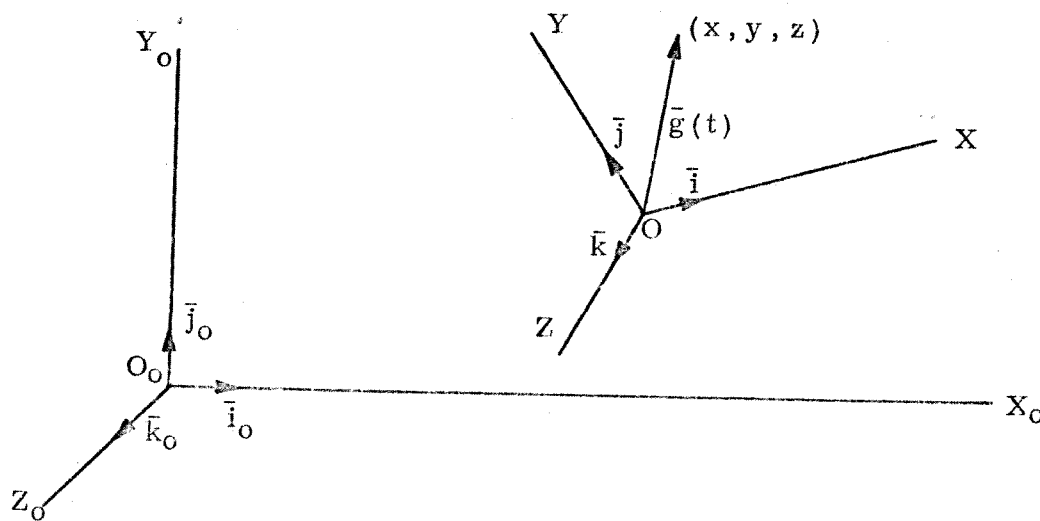


Fig. 27

The vector $\bar{g}$ can be written in component form

$$\bar{g} = \bar{i} x + \bar{j} y + \bar{k} z$$

Taking the derivative of $\bar{g}$, we have

$$\frac{d\bar{g}}{dt} = \frac{d}{dt} (\bar{i} x + \bar{j} y + \bar{k} z) = (\dot{\bar{i}} x + \dot{\bar{j}} y + \dot{\bar{k}} z) + (\bar{i} \dot{x} + \bar{j} \dot{y} + \bar{k} \dot{z})$$

Now $(\bar{i} \dot{x} + \bar{j} \dot{y} + \bar{k} \dot{z})$ is the time derivative of $\bar{g}$ measured by an observer stationed on the moving frame, and is defined by the relation

$$\dot{\bar{g}}_{\text{rel}} = \bar{i} \dot{x} + \bar{j} \dot{y} + \bar{k} \dot{z}$$

where $\dot{\bar{g}}_{\text{rel}}$ is the time derivative of $\bar{g}$ relative to the moving frame. Substitution yields

$$\frac{d\bar{g}}{dt} = \dot{\bar{g}}_{\text{rel}} + (\dot{\bar{i}} x + \dot{\bar{j}} y + \dot{\bar{k}} z) \quad (2.11)$$

To obtain expressions for $\dot{\bar{i}}$, $\dot{\bar{j}}$ and $\dot{\bar{k}}$, consider $\bar{i} \cdot \bar{i} = 1$. Differentiating, we have

$$\frac{d}{dt} (\bar{i} \cdot \bar{i}) = 2\bar{i} \cdot \dot{\bar{i}} = 0$$

Therefore, $\bar{i}$ is generally perpendicular to $\dot{\bar{i}}$. Similarly, $\bar{j} \perp \dot{\bar{j}}$ and $\bar{k} \perp \dot{\bar{k}}$. Since $\bar{i}$ is perpendicular to $\dot{\bar{i}}$, $\dot{\bar{i}}$ lies in a plane parallel to the Y-Z plane, and can therefore be written as a linear combination of the $\bar{j}$ and $\bar{k}$ unit vectors. Thus,

$$\dot{\bar{i}} = c \bar{j} + d \bar{k}$$

Similarly,

$$\dot{\bar{j}} = e \bar{i} + a \bar{k} \quad \text{and} \quad \dot{\bar{k}} = b \bar{i} + f \bar{j}$$

In the above expressions, the coefficients a , b , c , d , e , and f are generally functions of time.

Further, we recall that $\bar{i} = \bar{j} \times \bar{k}$, $\bar{j} = \bar{k} \times \bar{i}$, and $\bar{k} = \bar{i} \times \bar{j}$. Differentiating the above expression for $\bar{i}$ gives

$$\dot{\bar{i}} = \bar{j} \times \dot{\bar{k}} + \dot{\bar{j}} \times \bar{k}$$

and substituting the values of $\dot{\bar{j}}$ and $\dot{\bar{k}}$ found above, we have

$$\dot{\bar{i}} = \bar{j} \times (b\bar{i} + f\bar{j}) + (e\bar{i} + a\bar{k}) \times \bar{k}$$

$$\dot{\bar{i}} = -b\bar{k} + f(\bar{j} \times \bar{j}) - e\bar{j} + a(\bar{k} \times \bar{k})$$

$$\dot{\bar{i}} = -e\bar{j} - b\bar{k}$$

In similar fashion, we find that

$$\dot{\bar{j}} = -f\bar{k} - c\bar{i} \quad \text{and} \quad \dot{\bar{k}} = -d\bar{i} - a\bar{j}$$

Thus we have the following expressions for $\dot{\bar{i}}$, $\dot{\bar{j}}$, and $\dot{\bar{k}}$:

$$\dot{\bar{i}} = c\bar{j} + d\bar{k} = -e\bar{j} - b\bar{k}$$

$$\dot{\bar{j}} = e\bar{i} + a\bar{k} = -c\bar{i} - f\bar{k}$$

$$\dot{\bar{k}} = b\bar{i} + f\bar{j} = -d\bar{i} - a\bar{j}$$

from which, we see that

$$a = -f$$

$$b = -d$$

$$c = -e$$

and we now write

$$\dot{\bar{i}} = c \bar{j} - b \bar{k}$$

$$\dot{\bar{j}} = a \bar{k} - c \bar{i}$$

$$\dot{\bar{k}} = b \bar{i} - a \bar{j}$$

The vector $(\dot{\bar{i}} x + \dot{\bar{j}} y + \dot{\bar{k}} z)$ can now be written:

$$(\dot{\bar{i}} x + \dot{\bar{j}} y + \dot{\bar{k}} z) = (c \bar{j} - b \bar{k})x + (a \bar{k} - c \bar{i})y + (b \bar{i} - a \bar{j})z$$

$$= (bz - cy)\bar{i} + (cx - az)\bar{j} + (ay - bx)\bar{k}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a & b & c \\ x & y & z \end{vmatrix}$$

$$= \bar{\omega} \times \bar{g}$$

where $\bar{\omega}$ is a vector with components a , b , and c ; that is, $\bar{\omega} = a\bar{i} + b\bar{j} + c\bar{k}$.

We can now rewrite Eq. (2.11) as

$$\frac{d\bar{g}}{dt} = \dot{\bar{g}} = \dot{\bar{g}}_{\text{rel}} + \bar{\omega} \times \bar{g} \quad (2.12)$$

To see what physical interpretation can be assigned to the vector $\bar{\omega}$, consider $\bar{g}$ to be fixed in the moving frame. Then we have $\dot{\bar{g}}_{\text{rel}} = 0$. Now we are left with the expression

$$\frac{d\bar{g}}{dt} = \bar{\omega} \times \bar{g} \quad (2.13)$$

Suppose the moving frame is not rotating and is in translational motion with respect to the reference frame; that is, the X , Y , and Z axes maintain a constant orientation with respect to the X_0 , Y_0 , and Z_0 axes as the frame moves. Then the unit vectors $\bar{i}$, $\bar{j}$, and $\bar{k}$ remain constant in orientation as well as in magnitude, and $\dot{\bar{i}}$, $\dot{\bar{j}}$, and $\dot{\bar{k}}$ are zero. Since

$$\bar{\omega} \times \bar{g} = \dot{\bar{i}}x + \dot{\bar{j}}y + \dot{\bar{k}}z$$

then $\bar{\omega} \times \bar{g} = 0$ when the moving frame is not rotating with respect to the reference frame.

Now consider the case (Fig. 28) in which the moving frame is rotating and is not in translational motion with respect to the reference frame. Let the instantaneous angular velocity of the moving frame be $\bar{\Omega}$. Since $\bar{g}$ is fixed in the moving frame by assumption, it, too, possesses the angular velocity $\bar{\Omega}$ with respect to the reference frame. As the moving coordinate frame rotates, the tip of the vector $\bar{g}$ sweeps through an arc in space. The linear velocity of this point along the arc is given by

$$\frac{d\bar{g}}{dt}$$

But (paragraph 2.10) the linear velocity of a point is related to the angular velocity of its position vector by

$$\frac{d\bar{g}}{dt} = \bar{\Omega} \times \bar{g}$$

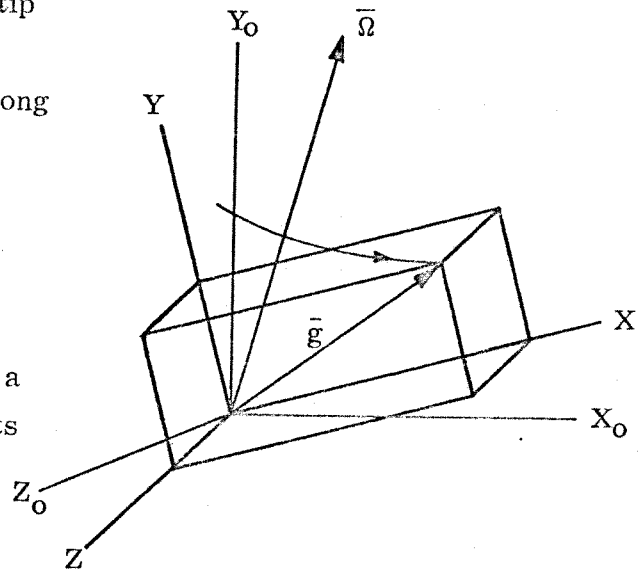


Fig. 28

Referring to Eq. (2.13), we see that the vector $\bar{\omega}$ is actually the angular velocity $\bar{\Omega}$ of the moving frame.

Thus, when two coordinate systems are in motion with respect to each other, the time derivative of a vector $\bar{g}$ measured in one frame is equal to the time derivative of the same vector measured in the other frame (i. e., $\dot{\bar{g}} = \dot{\bar{g}}_{\text{rel}}$) plus a term $\bar{\omega} \times \bar{g}$, where $\bar{\omega}$ is the angular velocity of the second frame with respect to the first. This relationship is stated in Eq. (2.12). (Since the designations reference frames and moving frames are arbitrarily assigned, we did not use them here. The stated relationship is a mutual one; either frame may serve as the reference frame.)

Now we may turn our attention to the problem originally posed: How can we extend our concepts of velocity and acceleration to noninertial reference frames?

In Fig. 29, we consider a reference frame (X_0, Y_0, Z_0) and a second frame (X, Y, Z) in motion with respect to it. The reference frame may or may not be an inertial frame.

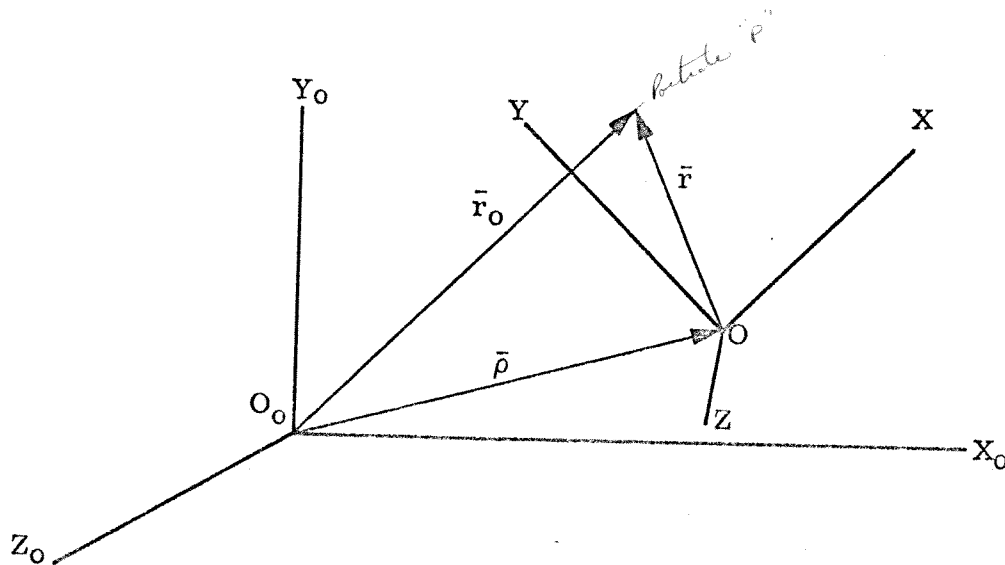


Fig. 29

An observer stationed on the moving frame measures the velocity of a particle P with respect to his coordinate system. Let this velocity be denoted $\bar{v}_{\text{rel}}$, the velocity of the particle relative to the moving coordinate system. We now wish to find the velocity of the particle with respect to the reference frame.

From Fig. 29, we have

$$\bar{r}_O = \bar{\rho} + \bar{r}$$

where $\bar{\rho}$ is the position vector of the origin of the moving frame, $\bar{r}$ is the position vector of P measured in the moving frame, and $\bar{r}_O$ is the position vector of P measured in the reference frame. Differentiating, we have

$$\frac{d\bar{r}_O}{dt} = \bar{v}_O = \dot{\bar{\rho}} + \dot{\bar{r}}$$

We have seen in Eq. (2.12), however, that

$$\dot{\bar{r}} = \bar{v}_{\text{rel}} + \bar{\omega} \times \bar{r}$$

where $\bar{\omega}$ is the angular velocity of the moving frame with respect to the reference frame. Thus,

$$\bar{v}_O = \dot{\bar{\rho}} + \bar{v}_{\text{rel}} + \bar{\omega} \times \bar{r} \quad (2.14)$$

Equation (2.14) expresses the relationship between the velocity of a particle relative to a reference frame and the velocity of the particle relative to a second frame in motion with respect to the reference frame.

To determine an expression for the acceleration of the particle relative to the reference frame in terms of its acceleration relative to the moving frame, we shall formally differentiate Eq. (2.14):

$$\bar{a}_o = \dot{\bar{v}}_o = \ddot{\bar{\rho}} + \dot{\bar{v}}_{rel} + \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times \dot{\bar{r}}$$

From Eq. (2.12), we have

$$\dot{\bar{r}} = \bar{v}_{rel} + \bar{\omega} \times \bar{r}$$

and

$$\dot{\bar{v}}_{rel} = \bar{a}_{rel} + \bar{\omega} \times \bar{v}_{rel}$$

where $\bar{a}_{rel}$ is the acceleration of the particle as measured by an observer stationed on the moving frame. Substitution of these values for $\dot{\bar{r}}$ and $\dot{\bar{v}}_{rel}$ yields

$$\begin{aligned}\bar{a}_o &= \ddot{\bar{\rho}} + \bar{a}_{rel} + \bar{\omega} \times \bar{v}_{rel} + \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times (\bar{v}_{rel} + \bar{\omega} \times \bar{r}) \\ \bar{a}_o &= \ddot{\bar{\rho}} + \bar{a}_{rel} + \dot{\bar{\omega}} \times \bar{r} + 2\bar{\omega} \times \bar{v}_{rel} + \bar{\omega} \times (\bar{\omega} \times \bar{r})\end{aligned}\quad (2.15)$$

We are particularly interested in the case where the reference frame is an inertial frame, for Eqs. (2.14) and (2.15) express the inertial velocity and acceleration of a particle, in terms of which, the equation of motion of the particle can be written

$$\begin{aligned}\bar{F} &= m\bar{a}_o = m\bar{a}_{rel} + m[\ddot{\bar{\rho}} + \dot{\bar{\omega}} \times \bar{r} + 2\bar{\omega} \times \bar{v}_{rel} + \bar{\omega} \times (\bar{\omega} \times \bar{r})] \\ m\bar{a}_{rel} &= \bar{F} - m[\ddot{\bar{\rho}} + \dot{\bar{\omega}} \times \bar{r} + 2\bar{\omega} \times \bar{v}_{rel} + \bar{\omega} \times (\bar{\omega} \times \bar{r})]\end{aligned}\quad (2.16)$$

An observer stationed on the moving frame measures the velocity of a particle P with respect to his coordinate system. Let this velocity be denoted $\bar{v}_{\text{rel}}$, the velocity of the particle relative to the moving coordinate system. We now wish to find the velocity of the particle with respect to the reference frame.

From Fig. 29, we have

$$\bar{r}_O = \bar{\rho} + \bar{r}$$

where $\bar{\rho}$ is the position vector of the origin of the moving frame, $\bar{r}$ is the position vector of P measured in the moving frame, and $\bar{r}_O$ is the position vector of P measured in the reference frame. Differentiating, we have

$$\frac{d\bar{r}_O}{dt} = \bar{v}_O = \dot{\bar{\rho}} + \dot{\bar{r}}$$

We have seen in Eq. (2.12), however, that

$$\dot{\bar{r}} = \bar{v}_{\text{rel}} + \bar{\omega} \times \bar{r}$$

where $\bar{\omega}$ is the angular velocity of the moving frame with respect to the reference frame. Thus,

$$\bar{v}_O = \dot{\bar{\rho}} + \bar{v}_{\text{rel}} + \bar{\omega} \times \bar{r} \quad (2.14)$$

Equation (2.14) expresses the relationship between the velocity of a particle relative to a reference frame and the velocity of the particle relative to a second frame in motion with respect to the reference frame.

To determine an expression for the acceleration of the particle relative to the reference frame in terms of its acceleration relative to the moving frame, we shall formally differentiate Eq. (2.14):

$$\bar{a}_o = \dot{\bar{v}}_o = \ddot{\bar{\rho}} + \dot{\bar{v}}_{rel} + \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times \dot{\bar{r}}$$

From Eq. (2.12), we have

$$\dot{\bar{r}} = \bar{v}_{rel} + \bar{\omega} \times \bar{r}$$

and

$$\dot{\bar{v}}_{rel} = \bar{a}_{rel} + \bar{\omega} \times \bar{v}_{rel}$$

where $\bar{a}_{rel}$ is the acceleration of the particle as measured by an observer stationed on the moving frame. Substitution of these values for $\dot{\bar{r}}$ and $\dot{\bar{v}}_{rel}$ yields

$$\begin{aligned}\bar{a}_o &= \ddot{\bar{\rho}} + \bar{a}_{rel} + \bar{\omega} \times \bar{v}_{rel} + \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times (\bar{v}_{rel} + \bar{\omega} \times \bar{r}) \\ \bar{a}_o &= \ddot{\bar{\rho}} + \bar{a}_{rel} + \dot{\bar{\omega}} \times \bar{r} + 2\bar{\omega} \times \bar{v}_{rel} + \bar{\omega} \times (\bar{\omega} \times \bar{r})\end{aligned}\quad (2.15)$$

We are particularly interested in the case where the reference frame is an inertial frame, for Eqs. (2.14) and (2.15) express the inertial velocity and acceleration of a particle, in terms of which, the equation of motion of the particle can be written

$$\begin{aligned}\bar{F} &= m\bar{a}_o = m\bar{a}_{rel} + m[\ddot{\bar{\rho}} + \dot{\bar{\omega}} \times \bar{r} + 2\bar{\omega} \times \bar{v}_{rel} + \bar{\omega} \times (\bar{\omega} \times \bar{r})] \\ m\bar{a}_{rel} &= \bar{F} - m[\ddot{\bar{\rho}} + \dot{\bar{\omega}} \times \bar{r} + 2\bar{\omega} \times \bar{v}_{rel} + \bar{\omega} \times (\bar{\omega} \times \bar{r})]\end{aligned}\quad (2.16)$$

To an observer in the moving frame, the particle would appear to be moving in accordance with the equation of motion

$$\bar{\mathbf{F}} + \bar{\mathbf{F}}_{\text{app}} = m\bar{\mathbf{a}}_{\text{rel}}$$

where $\bar{\mathbf{F}}$ is the true force acting on the particle, and $\bar{\mathbf{F}}_{\text{app}}$ is an apparent force also influencing the particle, and where

$$\bar{\mathbf{F}}_{\text{app}} = -m [\ddot{\bar{\rho}} + \dot{\bar{\omega}} \times \bar{\mathbf{r}} + 2\bar{\omega} \times \bar{\mathbf{v}}_{\text{rel}} + \bar{\omega} \times (\bar{\omega} \times \bar{\mathbf{r}})] \quad (2.17)$$

As an example of this apparent force, consider an inertial coordinate system fixed at the center of the earth (of course, since the earth is rotating around the sun, which is in turn in motion, the system is not truly inertial, but its motion has negligible effect on the problem under consideration): An observer is making observations of particle motion in a reference frame fixed to the earth with origin at the center of the earth, and this noninertial frame is rotating with respect to the inertial frame with angular velocity $\bar{\omega}_e$. Since the origins of both frames coincide, $\ddot{\bar{\rho}} = 0$. Further, assuming $\bar{\omega}_e$ to be constant in magnitude and direction, $\dot{\bar{\omega}}_e = 0$. Equation (2.16) thus reduces to

$$m\bar{\mathbf{a}}_{\text{rel}} = \bar{\mathbf{F}} - m [2\bar{\omega} \times \bar{\mathbf{v}}_{\text{rel}} + \bar{\omega} \times (\bar{\omega} \times \bar{\mathbf{r}})]$$

The apparent force $-2m\bar{\omega} \times \bar{\mathbf{v}}_{\text{rel}}$ is the Coriolis force, whose action is perpendicular to the relative velocity vector $\bar{\mathbf{v}}_{\text{rel}}$. The apparent force $-m\bar{\omega} \times (\bar{\omega} \times \bar{\mathbf{r}})$ is the centrifugal force whose action is outward from the earth's axis. The equation of motion written in the noninertial system must include these two apparent forces in addition to the true force $\bar{\mathbf{F}}$.

We have seen that, when we are interested in finding the motion of a particle with respect to a noninertial reference frame, the equation of motion can be written in two forms:

$$\bar{\mathbf{F}} + \bar{\mathbf{F}}_{\text{app}} = m\bar{\mathbf{a}}_{\text{rel}}$$

Here, the acceleration of the particle relative to the moving frame is obtained directly from solution of the equation of motion written in terms of the moving frame. It will be noted, however, that all the apparent forces appearing in Eq. (2.17) must be included as $\bar{\mathbf{F}}_{\text{app}}$.

$$\bar{\mathbf{F}} = m\bar{\mathbf{a}}_0$$

Here, the inertial acceleration of the particle is found directly from the true forces acting on the particle. Integration leads to the inertial velocity and inertial position of the particle at any instant of time. The position of the particle relative to the moving frame at any instant can be found from the inertial position by means of a coordinate transformation of the type described in Section 1.